Differential Equation Basics

General: Equations that contain an unknown function and one or more of its derivatives. The highest derivative in the equation is called its *order.* The function that satisfies the differential equation is called a *solution.*

Direction Field: If the differential equation can be expressed as $y' = F(x,y)$, it is called a *direction field.* It means, for every point on the x-y plane, a direction (slope – vector without magnitude) can be defined.

Separable Equations: A first-order differential equation which can be expressed as $\dfrac{dy}{dx} = g(x)f(y)$.

If $f(y) \neq 0$, this can be expressed as $\displaystyle\int \frac{1}{f(y)}\,dy = \int g(y)\,dx$ and therefore solved.

e.g. $\dfrac{dy}{dx} = x^2 y$, $\displaystyle\int \frac{dy}{y} = \int x^2 dx \quad (y \neq 0)$, $\ln|y| = \dfrac{x^3}{3} + C$,

$|y| = e^{\frac{x^3}{3}+C} = e^C e^{\frac{x^3}{3}}$, $y = Ae^{\frac{x^3}{3}} \quad (A = \pm e^C,\ A \neq 0,\ y \neq 0)$.

First order linear ODEs (Ordinary Differential Equations): $\dfrac{dy}{dx} + f(x)y = g(x)$.

Let $h(x) = e^{\int f(x)dx}$, called the *integrating factor,* then $\dfrac{d}{dx}h(x) = h(x)f(x)$.

Examine: $\dfrac{d}{dx}\Big(h(x)y\Big) = h(x)\dfrac{dy}{dx} + y\dfrac{d}{dx}h(x) = h(x)\dfrac{dy}{dx} + h(x)f(x)y = h(x)g(x)$.

Solve: $\dfrac{d}{dx}\Big(h(x)y\Big) = h(x)g(x)$.

e.g. $\dfrac{dy}{dx} + 3y = e^{-x}$. Let $h(x) = e^{3x}$. $\dfrac{d}{dx}(e^{3x}y) = e^{3x}e^{-x} = e^{2x}$.

$e^{3x}y = \dfrac{1}{2}e^{2x} + C$. $y = \dfrac{1}{2}e^{-x} + Ce^{-3x}$.

Exact ODEs: $H(x,y) = C$.

Consider: $dH(x,y) = \dfrac{\partial H}{\partial x}dx + \dfrac{\partial H}{\partial y}dy = 0$. $\dfrac{\partial H}{\partial x} + \dfrac{\partial H}{\partial y}\dfrac{dy}{dx} = 0$.

This is in the form of $F(x,y) + G(x,y)\dfrac{dy}{dx} = 0$, where $F = \dfrac{\partial H}{\partial x}$ and $G = \dfrac{\partial H}{\partial y}$.

Given that $\dfrac{\partial^2 H}{\partial y \partial x} = \dfrac{\partial^2 H}{\partial x \partial y}$ needs to hold true, $\dfrac{\partial F}{\partial y} = \dfrac{\partial G}{\partial x}$ is required.

An ODE of the form $F(x,y) + G(x,y)\dfrac{dy}{dx} = 0$ is *exact* if $\dfrac{\partial F}{\partial y} = \dfrac{\partial G}{\partial x}$.

An exact ODE can be solved by solving $F = \dfrac{\partial H}{\partial x}$ and $G = \dfrac{\partial H}{\partial y}$ for H, and the solution is $H(x,y) = C$.

e.g. $2xy + (x^2 + y^2)\dfrac{dy}{dx} = 0$. Solution: Let $F(x,y) = 2xy$ and $G(x,y) = x^2 + y^2$.

The ODE is exact as $\dfrac{\partial F}{\partial y} = 2x$ and $\dfrac{\partial G}{\partial x} = 2x$.

Solve $\dfrac{\partial H}{\partial x} = 2xy$, $H = x^2 y + C_1(y) \ldots (1)$ Solve $\dfrac{\partial H}{\partial y} = x^2 + y^2$, $H = x^2 y + \dfrac{1}{3}y^3 + C_2(x) \ldots (2)$

To find a common H: $C_1(y) = \dfrac{1}{3}y^3$, and $C_2(x) = 0$. $\therefore H(x,y) = x^2 y + \dfrac{1}{3}y^3$.

The solution is $3x^2 y + y^3 = A$.

Second order linear ODEs: $\dfrac{d^2y}{dx^2} + a\dfrac{dy}{dx} + by = f.$

The ODE is *homogeneous* if $f = 0$. i.e. $\dfrac{d^2y}{dx^2} + a\dfrac{dy}{dx} + by = 0.$

If $y1$ and $y2$ are two solutions for a homogeneous ODE, any linear combination $Ay_1 + By_2$ $(A, B \in \mathbf{R})$ is also a solution.

$$\frac{d^2}{dx^2}(Ay_1 + By_2) + a\frac{d}{dx}(Ay_1 + By_2) + b(Ay_1 + By_2) = A\left(\frac{d^2y_1}{dx^2} + a\frac{dy_1}{dx} + by_1\right) + B\left(\frac{d^2y_2}{dx^2} + a\frac{dy_2}{dx} + by_2\right) = 0.$$

If $y = e^{\lambda x}$, then $y' = \lambda y$ and $y'' = \lambda^2 y$, and the ODE becomes $\lambda^2 y + a\lambda y + by = 0$. As $y \neq 0$, $\lambda^2 + a\lambda + b = 0$.

Let λ_1 and λ_2 are the two solutions of $\lambda^2 y + a\lambda y + by = 0$, there are three cases:

Case 1: If λ_1 and λ_2 are two distinct real roots, the general solution to the ODE is $y = Ae^{\lambda_1 x} + Be^{\lambda_2 x}$.

Case 2: If λ_1 and λ_2 are equal (must be both real), the general solution to the ODE is $y = Ae^{\lambda x} + Bxe^{\lambda x}$.

Case 3: If λ_1 and λ_2 are two distinct complex roots $\alpha + \beta i$ and $\alpha - \beta i$ ($\alpha, \beta \in \mathbf{R}$ and $\beta \neq 0$), $y = Ce^{(\alpha+\beta i)x} + De^{(\alpha-\beta i)x}$.

By choosing complex numbers C and D, we can find the general solution that is real.

$$y = Ce^{\alpha x}e^{i\beta x} + De^{\alpha x}e^{-i\beta x} = e^{\alpha x}\big(C(\cos\beta x + i\sin\beta x) + D(\cos\beta x - i\sin\beta x)\big) = e^{\alpha x}\big((C+D)\cos\beta x + i(C-D)\sin\beta x\big)$$

The general solution is therefore $y = e^{\alpha x}\big(A\cos\beta x + B\sin\beta x\big)$.

(Note: This is by chooing $C = \frac{A}{2} - i\frac{B}{2}$ and $D = \frac{A}{2} + i\frac{B}{2}$.)

One type of non-homogeneous ODE is that f is a polynomial of degree P. i.e. $\dfrac{d^2y}{dx^2} + a\dfrac{dy}{dx} + by = f_P(x).$

If y_P is a solution, then the general solution is $y_H + y_P$, where y_H is the general solution of $\dfrac{d^2y}{dx^2} + a\dfrac{dy}{dx} + by = 0.$

Given that y_P must be of degree P, we can deduce that $\dfrac{dy}{dx} = \displaystyle\sum_{r=1}^{P} c_r r x^{r-1}$ and $\dfrac{d^2y}{dx^2} = \displaystyle\sum_{r=2}^{P} c_r r(r-1)x^{r-2}$.

We can solve for y_P by finding all zeros of polynomial $\displaystyle\sum_{r=2}^{P} c_r r(r-1)x^{r-2} + a\sum_{r=1}^{P} c_r r x^{r-1} + b\sum_{r=0}^{P} c_r x^r = f_P(x).$

e.g. $y'' - 5y' + 6y = 12x - 4$.

First solve $y'' - 5y' + 6y = 0$ (for y_H). $\lambda^2 - 5\lambda + 6 = 0$ has roots $\lambda = 2$ and $\lambda = 3$. $y_H = Ae^{2x} + Be^{3x}$.

Let $y_P = ax + b$, $y' = a$ and $y'' = 0$. So $0 - 5(a) + 6(ax + b) = 12x - 4$. $y_P = 2x + 1$.

Solution: $y = y_H + y_P = Ae^{2x} + Be^{3x} + 2x + 1$.

If f is not a polynomial, you may need to "guess" what y_P looks like.