Combined Uncertainty

Basics: Uncertainty is how much doubt about a measurement. e.g. If there is 95% of confidence that the length of a rod is between 9.95 m and 10.05 m, we say the uncertainty is 0.05 m, i.e. 10.00 ± 0.05 m.

Combined Uncertainty is the uncertainty of a value computed from other values or measurements. The formula to compute the Combined Uncertainty is

$$\Delta z = \sqrt{\sum_{k=1}^{n} \left(\Delta x_k \cdot \frac{\partial f}{\partial x_k} \right)^2} \qquad \text{where} \quad z = f(x_1, x_2, \cdots, x_n). \qquad \text{(Uncertainty is always positive.)}$$

This means that if the value of z is computed from other values or measurements $x_1, x_2, \cdots, x_n$ through function $f()$, then Δz, the uncertainty of z, can be computed through the above formula.

Sometimes the function $f()$ is implied: $$\boxed{\Delta z = \sqrt{\left(\Delta x_1 \cdot \frac{\partial z}{\partial x_1} \right)^2 + \left(\Delta x_2 \cdot \frac{\partial z}{\partial x_2} \right)^2 + \cdots + \left(\Delta x_n \cdot \frac{\partial z}{\partial x_n} \right)^2}\ .}$$

Addition: $z = \sum_{k=1}^{n} x_k = x_1 + x_2 + \cdots + x_n$.

$$\because \frac{\partial z}{\partial x_k} = 1\,, \qquad \therefore \Delta z = \sqrt{\sum_{k=1}^{n} (\Delta x_k)^2}\ . \qquad \text{i.e.} \ \boxed{\Delta z = \sqrt{\Delta x_1^2 + \Delta x_2^2 + \cdots + \Delta x_n^2}\ .}$$

Subtraction means $-x_k$ is among the terms, so $\frac{\partial z}{\partial x_k} = -1$ and $\left(\frac{\partial z}{\partial x_k} \right)^2 = 1$; the same formula will apply.

Index: $z = x^n$.

$$\because \frac{\partial z}{\partial x} = n x^{n-1} = \frac{nz}{x}\,, \qquad \therefore \Delta z = \sqrt{\left(\Delta x \cdot \frac{nz}{x} \right)^2} = \left| \frac{n \Delta x}{x} \cdot z \right| \ .$$

Divide both sides by $|z|$: $\boxed{\left| \frac{\Delta z}{z} \right| = \left| \frac{n \Delta x}{x} \right|\ .}$

Multiplication: $z = \prod_{k=1}^{n} x_k = x_1 \cdot x_2 \cdot \cdots \cdot x_n$.

$$\because \frac{\partial z}{\partial x_k} = \prod_{r=1, r \neq k}^{n} x_r = \frac{z}{x_k}\,, \qquad \therefore \Delta z = \sqrt{\sum_{k=1}^{n} \left(\Delta x_k \cdot \frac{z}{x_k} \right)^2}\ .$$

Divide both sides by $|z|$: $\boxed{\frac{\Delta z}{|z|} = \sqrt{\left(\frac{\Delta x_1}{x_1} \right)^2 + \left(\frac{\Delta x_2}{x_2} \right)^2 + \cdots + \left(\frac{\Delta x_n}{x_n} \right)^2}\ .}$

Division can be taken as x^{-1} and $\left(\frac{-1 \cdot \Delta x}{x} \right)^2 = \left(\frac{\Delta x}{x} \right)^2$; the same formula will apply.

Generally, if $z = \prod_{k=1}^{n} x_k^{a_k} = x_1^{a_1} \cdot x_2^{a_2} \cdot \cdots \cdot x_n^{a_n}$, $\frac{\Delta z}{|z|} = \sqrt{\left(\frac{a_1 \Delta x_1}{x_1} \right)^2 + \left(\frac{a_2 \Delta x_2}{x_2} \right)^2 + \cdots + \left(\frac{a_n \Delta x_n}{x_n} \right)^2}$.