MMAN3200 Notes

Linear Time Invariant (LTI):

$$\frac{d^n y(t)}{dt^n} = \sum_{i=1}^{n-1} a_i \cdot \frac{d^i y(t)}{dt^i} + \sum_{i=1}^{m} b_i \cdot \frac{d^i x(t)}{dt^i} + a_0 \cdot y(t) + b_0 \cdot x(t)$$

$\{a_i\}_{i=0}^{n-1}$, $\{b_i\}_{i=0}^{m}$ are independent of x and y (but technically can be time dependent as an ODE).
If in addition, they are independent of t (time) as well, the system is LTI.

Superposition Principle:

If $y_1(t)$ and $y_2(t)$ are the responses of $x_1(t)$ and $x_2(t)$ respectively, we know that $x(t) = x_1(t) + x_2(t)$ gives a response of $y(t) = y_1(t) + y_2(t)$. Generally, $x(t) = \sum_{k=1}^{N} c_k \cdot x_k(t)$ gives the response $y(t) = \sum_{k=1}^{N} c_k \cdot y_k(t)$.

Recall the Fourier Series: $f(x) = \sum_{n=-\infty}^{\infty} c_n \, e^{inx}$, where $c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, e^{-inx} \, dx$. (MATH2019)

$x(t) = \sum_{k=-\infty}^{\infty} c_k \cdot \phi_k(t)$, where $\phi_k(t) = e^{j \cdot \frac{2\pi}{T} \cdot k \cdot t}$. (It can be viewed that $x = \frac{2\pi}{T} t)$

To generalise to integrals of distributions: $x(t) = \int_{\mu=a}^{b} c(\mu) \cdot x_\mu(t) \cdot d\mu$,

which will give the response $y(t) = \int_{\mu=a}^{b} c(\mu) \cdot y_\mu(t) \cdot d\mu$, where $y_\mu(t)$ is the response of input $x_\mu(t)$.

Fourier Transformation:

$x(t) = \int_{-\infty}^{+\infty} x[\omega] \cdot e^{j\omega t} \cdot d\omega$, where $x[\omega]$ is the Fourier Transform of $x(t)$. It can be interpreted that $x[\omega]$ "amplifies" frequency ω in the continuous range from $-\infty$ to $+\infty$, and the result of the integral is the function $x(t)$.

Let $H(\omega)$ be the ratio of the input and response at frequency ω. The Frequency Response is $H(\omega) \cdot e^{j\omega t}$, which is the response to the "pure" harmonic function $e^{j\omega t}$ (without being "amplified" by $x[\omega]$).

The response to $x(t)$ is therefore $y(t) = \int_{-\infty}^{+\infty} x[\omega] \cdot H(\omega) \cdot e^{j\omega t} \cdot d\omega$

Convolution:

Run a function backward, shifted by t, then times it with another function and integrate the result over the domain.

$\boxed{(x * y)(t) = \int_{-\infty}^{+\infty} x(\tau) \cdot y(t-\tau) \, d\tau.}$ As $\int_{-\infty}^{+\infty} x(\tau) \cdot y(t-\tau) \, d\tau = \int_{-\infty}^{+\infty} y(\tau) \cdot x(t-\tau) \, d\tau$, $\boxed{x * y = y * x.}$

convolution theorem for laplace transform: $\mathcal{L}(x * y) = \mathcal{L}(x) \cdot \mathcal{L}(y)$.
If $z(t) = x(t) * y(t)$, then $z[\omega] = x[\omega] \cdot y[\omega]$. Also, if $z(t) = x(t) \cdot y(t)$, then $z[\omega] = x[\omega] * y[\omega]$.

Impulse response: Let $\delta(t)$ be M (magnitude) when $t \in [-\epsilon, +\epsilon]$, and zero otherwise, and also $\int_{-\infty}^{+\infty} \delta(t) \, dt = 1$.

Sampling: $\lim_{\epsilon \to 0} \int_{-\infty}^{+\infty} x(t) \cdot \delta(t-a) \, dt = x(a)$. (The limit is often implicit.) Since δ is even, $\delta(t-\tau) = \delta(\tau - t)$.

Putting t as τ and a as t, we will have $x(t) = \int_{-\infty}^{+\infty} x(\tau) \cdot \delta(t-\tau) \, d\tau = (x * \delta)(t)$.

The response to $x(t)$ will be $y(t) = (x * h)(t)$, where $h(t)$ is the response to impulse $\delta(t)$. Also $\mathcal{L}(y) = \mathcal{L}(x) \cdot \mathcal{L}(h)$.

Impedence: Input = Impedance × Output.

Electricity:

$v = R\,i,\quad V(s) = [R]\cdot J(s)\quad \ldots\quad$ Impedance is R.

$$v = \frac{1}{C}\int i\ dt,\quad V(s) = \left[\frac{1}{Cs}\right]\cdot J(s).$$

$$v = L\frac{di}{dt},\quad V(s) = [Ls]\cdot J(s).$$

Heat:

$$T_1 - T_2 = \left[-\frac{d}{KA}\right]\cdot q,\quad q = h_c A(T_1 - T_2),\quad \text{where the surface convective coefficient of heat transfer } h_c = -\frac{K}{d}.$$

$$q = C_t\,\frac{dT}{dt},\quad \text{where the thermal capacitance } C_t = \rho c V.$$

$$T = \frac{1}{C_t}\int q\ dt,\quad T(s) = \left[\frac{1}{C_t}\right]\cdot Q(s).$$

Fluid:

$P_1 - P_2 = [R_f]\ q.$

The bulk modulus $\beta \equiv \rho_0\cdot\dfrac{p-p_0}{\rho-\rho_0} = \rho_0\cdot\dfrac{\dot p}{\dot\rho}\ .\quad \dot p = \beta\cdot\dfrac{\dot\rho}{\rho_0} = \dfrac{\beta}{\rho_0}\cdot\dfrac{\dot m}{V},\quad \dot p = \left[\dfrac{\beta}{V\rho_0}\right]\dot m.$ ($\dot m$ is mass flow rate.)

Gas: $pV = mRT,\quad \dot p = \left[\dfrac{RT}{V}\right]\dot m.$

Inertia (liquid of density ρ through a tube of length l and cross section A with pressure p_1 and p_2 at two ends):

Volumetric flow rate $q = vA,\quad F = ma,\quad (p_1 - p_2)A = (\rho Al)\ddot l = (\rho Al)\dot v = (\rho Al)\dfrac{\dot q}{A} = \rho l\dot q.\quad \therefore \dot p = \left[\dfrac{\rho l}{A}\right]\dot q.$

Examples

In a mass spring system with an external force: $F = M\ddot x + kx.\quad F(s) = M(s^2X(s) - sx_0 - \dot{x_0}) + kX(s).$

If the system is initially at its neutral point at rest, $x_0 = 0$ and $\dot{x_0} = 0.\quad F(s) = X(s)(Ms^2 + k).$

The transfer function (Output/Input): $G(s) = \dfrac{X(s)}{F(s)} = \dfrac{1}{Ms^2 + k.}$

If F is a unit impulse $\delta(t), F(s) = \mathcal{L}\{\delta(t)\} = 1.\quad X(s) = \dfrac{1}{Ms^2 + k},$

$$x(t) = \mathcal{L}^{-1}\left\{\frac{1}{Ms^2 + k}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{\sqrt{Mk}}\cdot\frac{\sqrt{\frac{k}{M}}}{s^2 + \frac{k}{M}}\right\} = \frac{1}{\sqrt{Mk}}\cdot\sin\left(\sqrt{\frac{k}{M}}\cdot t\right).$$

Let $A = \dfrac{1}{M}$, $a = \sqrt{-\dfrac{k}{M}}$ and $b = -\sqrt{-\dfrac{k}{M}}.\quad ab = \dfrac{k}{M}.$

Non-dimensional form: $G(s) = \dfrac{\frac{1}{k}}{\frac{M}{k}s^2 + 1} = \dfrac{\frac{1}{k}}{\left(\sqrt{-\frac{M}{k}}s + 1\right)\left(-\sqrt{-\frac{M}{k}}s + 1\right)} = \dfrac{\frac{A}{ab}}{\left(\frac{s}{a} + 1\right)\left(\frac{s}{b} + 1\right)}.$

Normalised form: $G(s) = \dfrac{\frac{1}{M}}{\left(s + \sqrt{-\frac{k}{M}}\right)\left(s - \sqrt{-\frac{k}{M}}\right)} = \dfrac{A}{(s+a)(s+b)}.$