Q7 b) Given: $P(x_1, y_1)$ and $Q(x_2, y_2)$ are on ellipsis $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

Tangents on P and Q meet at $T(x_0, y_0)$, and $PT \parallel OQ$.

$O = (0, 0)$

$PT: \quad \dfrac{x_1 x}{a^2} + \dfrac{y_1 y}{b^2} = 1\,, \quad \text{Gradient } m_{PT} = -\dfrac{x_1 b^2}{y_1 a^2}$

$OQ: \quad \dfrac{y - 0}{x - 0} = m_{OQ} = m_{PT} = -\dfrac{x_1 b^2}{y_1 a^2}\,, \quad y = -\dfrac{x_1 b^2}{y_1 a^2} x\,, \quad \dfrac{x_1 x}{a^2} + \dfrac{y_1 y}{b^2} = 0$

Subsitute into $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ to find x_2 and y_2.

$$\frac{x^2}{a^2} + \frac{1}{b^2} \cdot \left(-\frac{x_1 b^2}{y_1 a^2} x\right)^2 = \frac{x^2}{a^2} + \frac{x_1^2 b^2}{y_1^2 a^4} x^2 = 1$$

$$a^2 y_1^2 x^2 + b^2 x_1^2 x^2 = a^4 y_1^2\,, \quad (a^2 y_1^2 + b^2 x_1^2) \cdot x^2 = a^4 y_1^2\,, \quad a^2 b^2 \cdot x^2 = a^4 y_1^2$$

$$x^2 = \frac{a^2 y_1^2}{b^2}$$

$x_2 = \dfrac{a y_1}{b}$ (assuming P in first quadrant and Q in fourth, similar for others)

$$y_2 = -\frac{x_1 b^2}{y_1 a^2} x_2 = -\frac{x_1 b^2}{y_1 a^2} \cdot \frac{a y_1}{b} = -\frac{b x_1}{a}$$

Perpendicular distance from P to OQ, using the general form of OQ:

$$d = \left| \frac{\frac{x_1}{a^2} x_1 + \frac{y_1}{b^2} y_1 + 0}{\sqrt{\left(\frac{x_1}{a^2}\right)^2 + \left(\frac{y_1}{b^2}\right)^2}} \right| = \left| \frac{\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}}{\sqrt{\left(\frac{x_1}{a^2}\right)^2 + \left(\frac{y_1}{b^2}\right)^2}} \right| = \left| \frac{1}{\sqrt{\frac{x_1^2}{a^4} + \frac{y_1^2}{b^4}}} \right|$$

$$= \frac{1}{\frac{1}{a^2 b^2} \sqrt{b^4 x_1^2 + a^4 y_1^2}} = \frac{a^2 b^2}{\sqrt{b^4 x_1^2 + a^4 y_1^2}}$$

$$\text{Area of } \triangle = \frac{1}{2} \cdot d \cdot OQ = \frac{1}{2} \cdot \frac{a^2 b^2}{\sqrt{b^4 x_1^2 + a^4 y_1^2}} \cdot \sqrt{x_2^2 + y_2^2}$$

$$= \frac{a^2 b^2}{2} \cdot \sqrt{\frac{x_2^2 + y_2^2}{b^4 x_1^2 + a^4 y_1^2}} = \frac{a^2 b^2}{2} \cdot \sqrt{\frac{\frac{a^2 y_1^2}{b^2} + \frac{b^2 x_1^2}{a^2}}{b^4 x_1^2 + a^4 y_1^2}}$$

$$= \frac{a^2 b^2}{2ab} \cdot \sqrt{\frac{a^4 y_1^2 + b^4 x_1^2}{b^4 x_1^2 + a^4 y_1^2}} = \frac{ab}{2}$$

$\therefore \triangle = \dfrac{ab}{2}$ which is independent of P