Year 12 Chapter 2B Question 5

In this text, $t = \tan \frac{1}{2}\theta$ (unless otherwise stated). As a result,

$\sin\theta = \frac{2t}{1+t^2}$, $\cos\theta = \frac{1-t^2}{1+t^2}$, $\tan\theta = \frac{2t}{1-t^2}$.

(a) $\cos\theta\,\left(\tan\theta - \tan\frac{1}{2}\theta\right) = \tan\frac{1}{2}\theta$

$$\text{LHS} = \frac{1-t^2}{1+t^2}\left(\frac{2t}{1-t^2} - t\right) = \frac{\cancel{1-t^2}}{1+t^2}\cdot\frac{2t - t(1-t^2)}{\cancel{1-t^2}} = \frac{2t - t + t^3}{1+t^2} = \frac{t\cancel{(1+t^2)}}{\cancel{1+t^2}} = t = \text{RHS}$$

(b) $\dfrac{1-\cos 2x}{\sin 2x} = \tan x$

Let $y = 2x$ and $t = \tan\frac{1}{2}y = \tan x$. $\therefore$ $\cos 2x = \cos y = \frac{1-t^2}{1+t^2}$ and $\sin 2x = \sin y = \frac{2t}{1+t^2}$.

$$\text{LHS} = \frac{1-\cos y}{\sin y} = \frac{1 - \frac{1-t^2}{1+t^2}}{\frac{2t}{1+t^2}} = \frac{\frac{(1+t^2)-(1-t^2)}{\cancel{1+t^2}}}{\frac{2t}{\cancel{1+t^2}}} = \frac{(1+t^2)-(1-t^2)}{2t} = \frac{2t^2}{2t} = t = \text{RHS}$$

(c) $\dfrac{1-\cos\theta}{1+\cos\theta} = \tan^2\frac{1}{2}\theta$

$$\text{LHS} = \frac{1 - \frac{1-t^2}{1+t^2}}{1 + \frac{1-t^2}{1+t^2}} = \frac{\frac{(1+t^2)-(1-t^2)}{\cancel{1+t^2}}}{\frac{(1+t^2)+(1-t^2)}{\cancel{1+t^2}}} = \frac{1+t^2-1+t^2}{1+t^2+1-t^2} = \frac{\cancel{2}t^2}{\cancel{2}} = t^2 = \text{RHS}$$

(d) $\dfrac{1+\csc\theta}{\cot\theta} = \dfrac{1+\tan\frac{1}{2}\theta}{1-\tan\frac{1}{2}\theta}$

$$\text{LHS} = \frac{1 + \frac{1+t^2}{2t}}{\frac{1-t^2}{2t}} = \frac{\frac{2t+(1+t^2)}{\cancel{2t}}}{\frac{1-t^2}{\cancel{2t}}} = \frac{1+2t+t^2}{1-t^2} = \frac{(1+t)^{\cancel{2}}}{\cancel{(1+t)}(1-t)} = \frac{1+t}{1-t} = \text{RHS}$$

(e) $\dfrac{\tan\theta\ \tan\frac{1}{2}\theta}{\tan\theta - \tan\frac{1}{2}\theta} = \sin\theta$

$$\text{LHS} = \frac{\frac{2t}{1-t^2}\cdot t}{\frac{2t}{1-t^2} - t} = \frac{\frac{2t^2}{\cancel{1-t^2}}}{\frac{2t-t(1-t^2)}{\cancel{1-t^2}}} = \frac{2t^2}{2t - t + t^3} = \frac{2t^{\cancel{2}}}{\cancel{t}(1+t^2)} = \frac{2t}{1+t^2} = \text{RHS}$$

(f) $\dfrac{\cos\theta + \sin\theta - 1}{\cos\theta - \sin\theta + 1} = \tan\frac{1}{2}\theta$

$$\text{LHS} = \frac{\frac{1-t^2}{1+t^2} + \frac{2t}{1+t^2} - 1}{\frac{1-t^2}{1+t^2} - \frac{2t}{1+t^2} + 1} = \frac{\frac{(1-t^2)+2t-(1+t^2)}{\cancel{1+t^2}}}{\frac{(1-t^2)-2t+(1+t^2)}{\cancel{1+t^2}}} = \frac{1-t^2+2t-1-t^2}{1-t^2-2t+1+t^2} = \frac{2t-2t^2}{2-2t} = \frac{\cancel{2(1-t)}\cdot t}{\cancel{2(1-t)}} = t = \text{RHS}$$

(g) $\dfrac{\tan 2\alpha + \cot\alpha}{\tan 2\alpha - \tan\alpha} = \cot^2\alpha$

Let $\beta = 2\alpha$ and $t = \tan\frac{1}{2}\beta = \tan\alpha$. $\therefore$ $\tan 2\alpha = \tan\beta = \frac{2t}{1-t^2}$.

$$\text{LHS} = \frac{\frac{2t}{1-t^2} + \frac{1}{t}}{\frac{2t}{1-t^2} - t} = \frac{\frac{2t^2+(1-t^2)}{\cancel{(1-t^2)t}}}{\frac{2t^2-t^2(1-t^2)}{\cancel{(1-t^2)t}}} = \frac{2t^2+1-t^2}{2t^2-t^2+t^4} = \frac{\cancel{1+t^2}}{t^2\cancel{(1+t^2)}} = \frac{1}{t^2} = \text{RHS}$$

(h) $\tan\left(\frac{1}{2}x + \frac{\pi}{4}\right) + \tan\left(\frac{1}{2}x - \frac{\pi}{4}\right) = 2\tan x$

$$\text{LHS} = \frac{\tan\frac{1}{2}x + \tan\frac{\pi}{4}}{1 - \tan\frac{1}{2}x\cdot\tan\frac{\pi}{4}} + \frac{\tan\frac{1}{2}x - \tan\frac{\pi}{4}}{1 + \tan\frac{1}{2}x\cdot\tan\frac{\pi}{4}} = \frac{\tan\frac{1}{2}x + 1}{1-\tan\frac{1}{2}x} + \frac{\tan\frac{1}{2}x - 1}{1+\tan\frac{1}{2}x}$$

$$= \frac{t+1}{1-t} + \frac{t-1}{1+t} = \frac{(1+t)^2-(1-t)^2}{(1-t)(1+t)} = \frac{2\cdot 2t}{1-t^2} = 2\tan x = \text{RHS}$$