Q3) Given $I_n = \int_0^{\frac{\pi}{4}} \tan^n x \, dx$ where $n \in I^+$ and $n \geq 3$

(a) Show that $I_n + I_{n-2} = \dfrac{1}{n-1}$

$$
\begin{aligned}
I_n + I_{n-2} &= \int_0^{\frac{\pi}{4}} \tan^n x \, dx + \int_0^{\frac{\pi}{4}} \tan^{n-2} x \, dx \\
&= \int_0^{\frac{\pi}{4}} \tan^{n-2} x \cdot (\tan^2 x + 1) \, dx \\
&= \int_0^{\frac{\pi}{4}} \tan^{n-2} x \cdot \sec^2 x \, dx \\
&= \int_0^{\frac{\pi}{4}} \tan^{n-2} x \;\, d(\tan x) \qquad [\, d(\tan x) = \sec^2 x \, dx \,] \\
&= \left[\frac{\tan^{n-1} x}{n-1} \right]_0^{\frac{\pi}{4}} = \frac{1}{n-1}
\end{aligned}
$$

(b) Hence evaluate I_5 and I_6

$$I_n = \frac{1}{n-1} - I_{n-2}$$

$$
\begin{aligned}
I_5 &= \frac{1}{5-1} - I_3 = \frac{1}{5-1} - \left(\frac{1}{3-1} - I_1 \right) \\
&= \frac{1}{4} - \frac{1}{2} + \int_0^{\frac{\pi}{4}} \tan x \, dx \\
&= -\frac{1}{4} - [\ln|\cos x|]_0^{\frac{\pi}{4}} = -\frac{1}{4} - \ln \frac{1}{\sqrt{2}} \\
&= \frac{1}{2} \ln 2 - \frac{1}{4}
\end{aligned}
$$

$$
\begin{aligned}
I_6 &= \frac{1}{6-1} - I_4 = \frac{1}{6-1} - \frac{1}{4-1} + I_2 \\
&= \frac{1}{5} - \frac{1}{3} + \int_0^{\frac{\pi}{4}} \tan^2 x \, dx \\
&= -\frac{2}{15} + [\tan x - x]_0^{\frac{\pi}{4}} = -\frac{2}{15} + 1 - \frac{\pi}{4} \\
&= \frac{13}{15} - \frac{\pi}{4}
\end{aligned}
$$