Ex 3 c) Find $\displaystyle\int \frac{dx}{a+b\cos x}$

$$\text{Let} \quad t=\tan\frac{x}{2}\,, \quad k=\sqrt{\left|\frac{a+b}{a-b}\right|}$$

$$\begin{aligned}
I &= \int \frac{\frac{2}{1+t^2}\,dt}{a+b\frac{1-t^2}{1+t^2}} \\
&= \int \frac{2\,dt}{a\,(1+t^2)+b(1-t^2)} \\
&= \int \frac{2\,dt}{(a-b)\,t^2+(a+b)} \qquad \ldots(1)
\end{aligned}$$

No solution when $a=0$ and $b=0$. Four cases left:
1) $a=b\neq 0$, 2) $a=-b\neq 0$, 3) $|a|>|b|$ and 4) $|b|>|a|$

Case 1: $a=b\neq 0$

$$\begin{aligned}
I &= \int \frac{2\,dt}{(a-b)\,t^2+(a+b)} \qquad \text{from (1)} \\
&= \int \frac{2\,dt}{a+b} = \int \frac{\cancel{2}}{\cancel{2}a}\,dt \\
&= \frac{1}{a}t+C = \frac{1}{a}\tan\frac{x}{2}+C
\end{aligned}$$

Case 2: $a=-b\neq 0$

$$\begin{aligned}
I &= \int \frac{2\,dt}{(a-b)\,t^2+(a+b)} \qquad \text{from (1)} \\
&= \int \frac{2\,dt}{(a-b)\,t^2} = \int \frac{\cancel{2}}{\cancel{2}at^2}\,dt \\
&= \frac{1}{a}\cdot\frac{-1}{t}+C = \frac{-1}{a\tan\frac{x}{2}}+C \\
&= -\frac{1}{a}\cot\frac{x}{2}+C
\end{aligned}$$

Case 3: $|a| > |b|$

ie $\quad (a+b)(a-b) = a^2 - b^2 > 0$

$\therefore$ $(a+b)$ and $(a-b)$ are of the same sign.

$$k = \sqrt{\frac{a+b}{a-b}}\,, \quad k^2 = \frac{a+b}{a-b}$$

$$\begin{aligned}
I &= \int \frac{2\,dt}{(a-b)\,t^2 + (a+b)} \qquad \text{from (1)} \\
&= \frac{2}{a-b} \int \frac{dt}{t^2 + \frac{a+b}{a-b}} = \frac{2}{a-b} \int \frac{dt}{t^2 + k^2} \\
&= \frac{2}{a-b} \cdot \frac{1}{k} \cdot \tan^{-1}\left(\frac{t}{k}\right) + C \\
&= \frac{2}{a-b} \cdot \sqrt{\frac{a-b}{a+b}} \cdot \tan^{-1}\left(\sqrt{\frac{a-b}{a+b}} \tan\frac{x}{2}\right) + C
\end{aligned}$$

Note: $\dfrac{2}{a-b} \cdot \sqrt{\dfrac{a-b}{a+b}} \neq \dfrac{2}{\sqrt{a-b}\sqrt{a+b}}$ (LHS can be negative but RHS is always positive.)

Case 4: $|b| > |a|$:

ie $\quad (b+a)(b-a) = b^2 - a^2 > 0$

$\therefore$ $(b+a)$ and $(b-a)$ are of the same sign.

$$k = \sqrt{\frac{b+a}{b-a}}\,, \quad k^2 = \frac{b+a}{b-a}$$

$$\begin{aligned}
I &= \int \frac{2\,dt}{(a-b)\,t^2 + (a+b)} \qquad \text{from (1)} \\
&= \int \frac{2\,dt}{(b+a) - (b-a)\,t^2} \\
&= \frac{1}{b-a} \int \frac{2\,dt}{\frac{b+a}{b-a} - t^2} = \frac{1}{b-a} \int \frac{2\,dt}{k^2 - t^2} \\
&= \frac{1}{b-a} \cdot \frac{1}{k} \cdot \int \frac{1}{k-t} + \frac{1}{k+t}\,dt \\
&= \frac{1}{b-a} \cdot \sqrt{\frac{b-a}{b+a}} \cdot \ln\left|\frac{\sqrt{\frac{b+a}{b-a}} + \tan\frac{x}{2}}{\sqrt{\frac{b+a}{b-a}} - \tan\frac{x}{2}}\right| + C
\end{aligned}$$

Note: $\dfrac{1}{b-a} \cdot \sqrt{\dfrac{b-a}{b+a}} \neq \dfrac{1}{\sqrt{b-a}\sqrt{b+a}}$ (LHS can be negative but RHS is always positive.)

In conclusion:

When $|a| > |b|$,

$$\int \frac{dx}{a + b\cos x} = \frac{2}{a-b} \cdot \sqrt{\frac{a-b}{a+b}} \cdot \tan^{-1}\left(\sqrt{\frac{a-b}{a+b}} \tan \frac{x}{2}\right) + C$$

When $|b| > |a|$,

$$\int \frac{dx}{a + b\cos x} = \frac{1}{b-a} \cdot \sqrt{\frac{b-a}{b+a}} \cdot \ln\left|\frac{\sqrt{\frac{b+a}{b-a}} + \tan \frac{x}{2}}{\sqrt{\frac{b+a}{b-a}} - \tan \frac{x}{2}}\right| + C$$