Ex 3 b) Find $\displaystyle\int \frac{dx}{a+b\sin x}$

Let $t=\tan\dfrac{x}{2}$, $u=at+b \quad (du=adt)$, $k=\sqrt{|a^2-b^2|}$

$$I=\int \frac{\frac{2}{1+t^2}\,dt}{a+b\frac{2t}{1+t^2}}=\int \frac{2\,dt}{a\left(1+t^2\right)+2bt} \qquad \ldots(1)$$

$$=\int \frac{2a\,dt}{a^2t^2+2abt+a^2}=\int \frac{2a\,dt}{\left(at+b\right)^2-b^2+a^2}=\int \frac{2\,du}{u^2-b^2+a^2} \qquad \ldots(2)$$

No solution when $a=0$ and $b=0$. Three cases left:

1) $|a|=|b|\neq 0$, 2) $|a|>|b|$ and 3) $|b|>|a|$

Case 1: $|a|=|b|\neq 0$

$$I=\int \frac{2du}{u^2}=\frac{-2}{u}+C=\frac{-2}{a\tan\frac{x}{2}+b}+C$$

Case 2: $|a|>|b| \quad \therefore\ k=\sqrt{a^2-b^2}$, $a^2-b^2=k^2$

$$I=\int \frac{2\,du}{u^2-b^2+a^2} \qquad \text{from (2)}$$

$$=\int \frac{2\,du}{u^2+k^2}=\frac{2}{k}\ \tan^{-1}\frac{u}{k}+C$$

$$=\frac{2}{\sqrt{a^2-b^2}}\ \tan^{-1}\frac{a\tan\frac{x}{2}+b}{\sqrt{a^2-b^2}}+C$$

Case 3a: $|b|>|a|>0 \quad \therefore\ k=\sqrt{b^2-a^2}$, $b^2-a^2=k^2$

$$I=\int \frac{2\,du}{u^2-b^2+a^2} \qquad \text{from (2)}$$

$$=\int \frac{2\,du}{u^2-k^2}=\frac{1}{k}\int \frac{1}{u-k}-\frac{1}{u+k}\,du$$

$$=\frac{1}{k}\ln\left|\frac{u-k}{u+k}\right|+C$$

$$=\frac{1}{\sqrt{b^2-a^2}}\ln\left|\frac{a\tan\frac{x}{2}+b-\sqrt{b^2-a^2}}{a\tan\frac{x}{2}+b+\sqrt{b^2-a^2}}\right|+C$$

Case 3b: $|b|>|a|=0$

$$I=\int \frac{2\,dt}{a\left(1+t^2\right)+2bt} \qquad \text{from (1)}$$

$$=\int \frac{\not{2}}{\not{2}bt}\,dt=\frac{\ln\left|\tan\frac{x}{2}\right|}{b}+C$$