Ex 3 a) Find $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{1 + k\cos x}$

$$\text{Let } t = \tan\frac{x}{2}$$

$$\text{When } x = \frac{\pi}{2}\,,\quad t = 1$$

$$I = \int_0^1 \frac{\frac{2}{1+t^2}\,dt}{1 + k\frac{1-t^2}{1+t^2}}$$

$$= \int_0^1 \frac{2\,dt}{(1+t^2) + k\,(1-t^2)}$$

$$k \neq -1\,,\quad \text{as}\quad t \in \left[0, \frac{\pi}{2}\right]$$

$$\text{When } k = 1\,,\quad I = \int_0^1 dt = 1$$

$$\text{When } -1 < k < 1,\quad 1-k > 0 \quad \text{and} \quad 1+k > 0$$

$$I = \int_0^1 \frac{2\,dt}{(1+k) + (1-k)\,t^2}$$

$$= 2 \cdot \frac{1}{\sqrt{1+k}} \cdot \frac{1}{\sqrt{1-k}} \left[\tan^{-1}\left(\frac{\sqrt{1-k}}{\sqrt{1+k}}t\right)\right]_0^1$$

$$= \frac{2}{\sqrt{1-k^2}}\ \tan^{-1}\sqrt{\frac{1-k}{1+k}}$$

In conclusion, when $|k| < 1$,

$$I = \frac{2}{\sqrt{1-k^2}}\ \tan^{-1}\sqrt{\frac{1-k}{1+k}}$$

From above, $\quad I = \displaystyle\int_0^1 \frac{2\,dt}{(1+t^2)+k\,(1-t^2)}$

When $k>1, \quad k-1>0 \quad$ and $\quad k+1>0$

$$
\begin{aligned}
I &= \int_0^1 \frac{2\,dt}{(k+1)-(k-1)\,t^2} \\
&= \frac{1}{\sqrt{k+1}} \int_0^1 \frac{1}{\sqrt{k+1}+\sqrt{k-1}t} + \frac{1}{\sqrt{k+1}-\sqrt{k-1}t}\,dt \\
&= \frac{1}{\sqrt{k+1}} \cdot \frac{1}{\sqrt{k-1}} \left[\ln\left|\sqrt{k+1}+\sqrt{k-1}t\right| - \ln\left|\sqrt{k+1}-\sqrt{k-1}t\right| \right]_0^1 \\
&= \frac{1}{\sqrt{k^2-1}} \left[\ln\left| \frac{\sqrt{k+1}+\sqrt{k-1}t}{\sqrt{k+1}-\sqrt{k-1}t} \right| \right]_0^1 \\
&= \frac{1}{\sqrt{k^2-1}} \left[\ln\left| \frac{\sqrt{k+1}+\sqrt{k-1}}{\sqrt{k+1}-\sqrt{k-1}} \right| - \ln\left| \frac{\sqrt{k+1}+0}{\sqrt{k+1}-0} \right| \right] \\
&= \frac{1}{\sqrt{k^2-1}} \ln\left| \frac{\sqrt{k+1}+\sqrt{k-1}}{\sqrt{k+1}-\sqrt{k-1}} \right|
\end{aligned}
$$

When $k<-1, \quad -1-k>0 \quad$ and $\quad 1-k>0$

$$
\begin{aligned}
I &= \int_0^1 \frac{2\,dt}{(1-k)\,t^2-(-1-k)} \\
&= \frac{1}{\sqrt{-1-k}} \int_0^1 \frac{1}{\sqrt{1-k}t-\sqrt{-1-k}} - \frac{1}{\sqrt{1-k}t+\sqrt{-1-k}}\,dt \\
&= \frac{1}{\sqrt{-1-k}} \cdot \frac{1}{\sqrt{1-k}} \left[\ln\left|\sqrt{1-k}t+\sqrt{-1-k}\right| - \ln\left|\sqrt{1-k}t-\sqrt{-1-k}\right| \right]_0^1 \\
&= \frac{1}{\sqrt{(-k)^2-1}} \left[\ln\left| \frac{\sqrt{1-k}t+\sqrt{-1-k}}{\sqrt{1-k}t-\sqrt{-1-k}} \right| \right]_0^1 \\
&= \frac{1}{\sqrt{k^2-1}} \left[\ln\left| \frac{\sqrt{1-k}+\sqrt{-1-k}}{\sqrt{1-k}-\sqrt{-1-k}} \right| - \ln\left| \frac{0+\sqrt{-1-k}}{0-\sqrt{-1-k}} \right| \right] \\
&= \frac{1}{\sqrt{k^2-1}} \ln\left| \frac{\sqrt{1-k}+\sqrt{-1-k}}{\sqrt{1-k}-\sqrt{-1-k}} \right|
\end{aligned}
$$

In conclusion, when $|k|>1$,

$$
= \frac{1}{\sqrt{k^2-1}} \ln\left| \frac{\sqrt{|k|+1}+\sqrt{|k|-1}}{\sqrt{|k|+1}-\sqrt{|k|-1}} \right|
$$