Eg2 (c) Show that $\int_0^\pi x\cos^2 x\,dx = \frac{\pi^2}{4}$

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\begin{aligned}
&\int_0^\pi x\cos^2 x\,dx \\
&= \int_0^\pi (\pi - x)\cos^2(\pi - x)\ dx \\
&= \int_0^\pi (\pi - x)(-\cos x)^2\ dx \\
&= \int_0^\pi \left(\pi\cos^2 x - x\cos^2 x\right)\ dx \\
&= \int_0^\pi \pi\cos^2 x\,dx - \int_0^\pi x\cos^2 x\,dx
\end{aligned}
$$

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\begin{aligned}
\therefore\quad 2\int_0^\pi x\cos^2 x\,dx &= \int_0^\pi \pi\cos^2 x\,dx \\
&= \int_0^{\frac{\pi}{2}} \pi\cos^2 x\,dx + \int_{\frac{\pi}{2}}^\pi \pi\cos^2 x\,dx \\
&= \pi\int_0^{\frac{\pi}{2}} \cos^2 x\,dx + \pi\int_0^{\frac{\pi}{2}} \cos^2\left(x + \frac{\pi}{2}\right)\ dx \\
&= \pi\int_0^{\frac{\pi}{2}} \cos^2 x\,dx + \pi\int_0^{\frac{\pi}{2}} (-\sin x)^2\ dx \\
&= \pi\int_0^{\frac{\pi}{2}} \left(\cos^2 x + \sin^2 x\right)\ dx \\
&= \pi\big[x\big]_0^{\frac{\pi}{2}} \\
&= \frac{\pi}{2}
\end{aligned}
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\therefore\quad \int_0^\pi x\cos^2 x\,dx = \frac{\pi}{4}
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