Ex 2. Under appropriate laboratory conditions the temperature $T°C$ of a beaker of a chemical solution and the temperature $S°C$ of a surrounding vat of cooler water satisfy according to Newton's Law of cooling:

$$\frac{dT}{dt} = -k(T - S) \qquad \text{where } k > 0\,. \tag{1}$$

$$\frac{dS}{dt} = 0.75k(T - S) \tag{2}$$

(a) Show that $\frac{3}{4} \cdot \frac{dT}{dt} + \frac{dS}{dt} = 0$ and hence deduce the result for $\frac{3}{4}T + S$.

Solution: $LHS = \frac{3}{4} \cdot \frac{dT}{dt} + \frac{dS}{dt} = -\frac{3}{4}k(T - S) + 0.75k(T - S) = 0$

$$\int_0^t \left(\frac{3}{4}\frac{dT}{dt} + \frac{dS}{dt} \right) \, dt = 0$$

$$\left[\frac{3}{4}T + S\right]_0^t = \left[\frac{3}{4}T + S\right] - \left[\frac{3}{4}T_0 + S_0\right] = 0$$

$$\frac{3}{4}T + S = \frac{3}{4}T_0 + S_0$$

(b) Find an expression for $\frac{dT}{dt}$ in terms of T, and

hence show that $T = \frac{4}{7}C + Ae^{-\frac{7}{4}kt}$, where C and A are constants,

satisfies this differential equation for any constant A.

Solution: From (a) $\frac{dS}{dt} = -\frac{3}{4}\frac{dT}{dt}$

$$\frac{d^2T}{dt^2} = \frac{d}{dt}\left(\frac{dT}{dt}\right) = \frac{d}{dt}\left[-k(T - S)\right] = -k\left(\frac{dT}{dt} - \frac{dS}{dt}\right) = -k\left(\frac{dT}{dt} + \frac{3}{4}\frac{dT}{dt}\right) = -\frac{7}{4}k \cdot \frac{dT}{dt}$$

$$\frac{dT}{dt} = \int \left(-\frac{7}{4}k \cdot \frac{dT}{dt}\right) \, dt = -k\left(\frac{7}{4}T - C\right) = -\frac{7}{4}k\left(T - \frac{4}{7}C\right)$$

$$\therefore \quad \frac{dT}{dt} = -\frac{7}{4}k\left(T - \frac{4}{7}C\right)$$

Given $T = \frac{4}{7}C + A\, e^{-\frac{7}{4}kt}$,

$$\frac{dT}{dt} = A\frac{d}{dt}e^{-\frac{7}{4}kt} = -\frac{7}{4}kA\, e^{-\frac{7}{4}kt} = -\frac{7}{4}k\left(T - \frac{4}{7}C\right) \, ,$$ which satisfies the differential equation.

(c) …

Ex 1. A cup of coffee cools at a rate proportional to the difference between its temperatures T_c and that of its surroundings. In winter, the room temperature is $15°C$ and I must wait 10 minutes for my coffee to cool from $90°C$ to $50°C$.

(a) Explain why: $\dfrac{dT_c}{dt} = -k(T_c - 15)$.

Solution: Newton's Law of Cooling ... duh! $(k > 0)$

(b) Show that: $T_c = 15 + 75\left(\dfrac{7}{15}\right)^{\frac{t}{10}}$

Solution: Based on the "duh" rule: $T_c = 15 + A\, e^{-kt}$

Given $A = 90 - 15 = 75$ and when $t = 10$,

$$T_c = 15 + 75\, e^{-10k} = 50\,, \quad e^{-10k} = \frac{35}{75} = \frac{7}{15}\,,$$

$$e^{-kt} = \left(e^{-10k}\right)^{\frac{t}{10}} = \left(\frac{7}{15}\right)^{\frac{t}{10}}\,,$$

$$\therefore\ T_c = 15 + A\, e^{-kt} = 15 + 75\left(\frac{7}{15}\right)^{\frac{t}{10}}$$

(c) How long must I wait, in summer, when the room temperature will be $25°C$ to cool to $50°C$?

Solution: So all the 15 is now 25, and $A = 90 - 25 = 65$.

$$T_c = 25 + 65\, e^{-kt} = 50\,, \quad e^{-kt} = \frac{25}{65} = \frac{5}{13}\,, \quad \left(\frac{7}{15}\right)^{\frac{t}{10}} = \frac{5}{13}$$

$$t = 10 \cdot \frac{\ln \frac{5}{13}}{\ln \frac{7}{15}} = 12.54 \text{ minutes} = 12:32 \text{ minutes}$$