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Q13

$$T = \frac{2V\sin\alpha}{g}, \qquad X = \frac{2V^2\sin\alpha\cos\alpha}{g}, \qquad u = V\cos\alpha - \frac{gd}{2V\sin\alpha}, \qquad gd = \frac{V^2}{2\sqrt{3}}.$$

iv) Show that no $0 \le \alpha \le \frac{\pi}{2}$ can satisfy $u > \dfrac{V}{\sqrt{3}}$.

Solution:

$$u = V\cos\alpha - \frac{V^2}{4\sqrt{3}\cdot V\sin\alpha} = V\left(\cos\alpha - \frac{1}{4\sqrt{3}\sin\alpha}\right).$$

$$\frac{du}{d\alpha} = V\left(-\sin\alpha + \frac{\sqrt{3}\cos\alpha}{12\sin^2\alpha}\right) = 0.$$

$$12\sin^3\alpha = \sqrt{3}\cos\alpha, \quad 144\sin^6\alpha = 3\cos^2\alpha, \quad 48\sin^6\alpha = 1-\sin^2\alpha, \quad 48 = \frac{1}{\sin^6\alpha} - \frac{1}{\sin^4\alpha},$$

$$x^3 - x^2 - 48 = 0 \text{ where } x = \frac{1}{\sin^2\alpha}, \quad (x^3-4x^2)+(3x^2-12x)+(12x-48)=0, \quad x^2(x-4)+3x(x-4)+12(x-4)=0,$$

$(x-4)(x^2+3x+12)=0, \quad x=4$ (The other factor has no real solutions.)

$$\frac{1}{\sin^2\alpha} = 4, \quad \sin^2\alpha = \frac{1}{4}, \quad \sin\alpha = \frac{1}{2} \ \left(0 < \alpha < \frac{\pi}{2}\right), \quad \alpha = \frac{\pi}{6}.$$

When $\alpha = 0$, $u \to -\infty$. When $\alpha = \dfrac{\pi}{6}$, $u = V\left(\dfrac{\sqrt{3}}{2} - \dfrac{1}{2\sqrt{3}}\right) = \dfrac{V}{\sqrt{3}}$ (max). When $\alpha = \dfrac{\pi}{2}$, $u = -\dfrac{V}{4\sqrt{3}}$.

This means u is at most equal to $\dfrac{V}{\sqrt{3}}$, never greater than.

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v) Since there is only one stationery point $u = \dfrac{V}{\sqrt{3}}$, which is a maximum in $\left[0, \dfrac{\pi}{2}\right]$, there are precisely two values of α to yield a $u < \dfrac{V}{\sqrt{3}}$, which correspond to two distances to hit at.

Q17

$$y = h + x\tan\alpha - x^2\frac{g}{2V^2\cos^2\alpha}.$$

d) Show that $\tan\alpha \ge \dfrac{Rh}{(R+r)r}$.

When $x = R$, $y = h + R\tan\alpha - R^2\dfrac{g}{2V^2\cos^2\alpha} \ge h$, $\quad \tan\alpha \ge R\dfrac{g}{2V^2\cos^2\alpha} \dots (1)$

When $x = R+r$, $y = h + (R+r)\tan\alpha - (R+r)^2\dfrac{g}{2V^2\cos^2\alpha} = 0$, $\quad \dfrac{g}{2V^2\cos^2\alpha} = \dfrac{h+(R+r)\tan\alpha}{(R+r)^2} \dots (2)$

Sub (2) into (1): $\quad \tan\alpha \ge \dfrac{Rh + R(R+r)\tan\alpha}{(R+r)^2}$, $\quad (R+r)^2\tan\alpha \ge Rh + R(R+r)\tan\alpha$,

$$(R+r-R)(R+r)\tan\alpha \ge Rh, \quad \tan\alpha \ge \frac{Rh}{(R+r)r}.$$

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e) $V \le 50$, $\quad \dfrac{gR}{2\sin\alpha\cos\alpha} \le V^2 \le 2500$, $\quad \sin 2\alpha \ge \dfrac{gR}{2500} = \dfrac{10\times 80}{2500} = 0.32$, $\quad 2\alpha = 0.32573$ or $\pi - 0.32573$.

We take the larger angle for a smaller r. So $\alpha = 1.408$, $\quad \tan\alpha \approx 6.0857 = \dfrac{Rh}{(R+r)r} = \dfrac{80 \times 1}{(80+r)r}$, $\quad r^2 + 80r - 13.146 = 0$

Solve for the smaller value of $r = 0.164$.