Q17

$$\int \frac{dx}{x\sqrt{a^2+x^2}}.$$

Solution:

Let $u = \sqrt{a^2+x^2}$, then $du = \frac{x}{\sqrt{a^2+x^2}}dx$. Please note that $a < u$ and $x^2 = u^2 - a^2$.

$$I = \int \frac{dx}{x\sqrt{a^2+x^2}} = \int \frac{du}{x^2} = \int \frac{du}{u^2-a^2} = \frac{1}{2a}\int \frac{1}{u-a} - \frac{1}{u+a}\,du = \frac{1}{2a}\left[\log(u-a) - \log(u+a)\right] + C = \frac{1}{2a}\log\left(\frac{u-a}{u+a}\right) + C.$$

Now you can convert this into different forms. To follow the answer, here is one way.

$$I = \frac{1}{2a}\log\left(\frac{u-a}{u+a}\right) + C = -\frac{1}{2a}\log\left(\frac{u+a}{u-a}\right) + C = -\frac{1}{2a}\log\left(\frac{u+a}{u-a}\cdot\frac{u+a}{u+a}\right) + C = -\frac{1}{2a}\log\left(\frac{(u+a)^2}{u^2-a^2}\right) + C$$
$$= -\frac{1}{a}\log\left(\frac{u+a}{x}\right) + C = -\frac{1}{a}\log\left(\frac{\sqrt{a^2+x^2}+a}{x}\right) + C.$$

Q15

$$\int \frac{1+x}{\sqrt{1-x-x^2}}dx.$$

Solution:

$$I = \int \frac{1+x}{\sqrt{1-x-x^2}}dx = \int \frac{1+x}{\sqrt{\frac{5}{4} - \left(\frac{1}{4}+x+x^2\right)}}dx = \int \frac{\frac{1}{2} + (\frac{1}{2}+x)}{\sqrt{\left(\frac{\sqrt{5}}{2}\right)^2 - \left(\frac{1}{2}+x\right)^2}}dx.$$

Let $u = \frac{1}{2} + x$ and $a = \frac{\sqrt{5}}{2}$.

$$I = \frac{1}{2}\int \frac{1}{\sqrt{a^2-u^2}}dx + \int \frac{u}{\sqrt{a^2-u^2}}dx = \frac{1}{2}\sin^{-1}\frac{u}{a} - \frac{1}{2}\int \frac{dv}{\sqrt{v}} \qquad \text{where } v = a^2 - u^2 = 1 - x - x^2,\ dv = -2udu.$$

$$= \frac{1}{2}\sin^{-1}\left(\frac{\frac{1}{2}+x}{\frac{\sqrt{5}}{2}}\right) - \frac{1}{2}\int \frac{dv}{\sqrt{v}} = \frac{1}{2}\sin^{-1}\frac{1+2x}{\sqrt{5}} - \sqrt{1-x-x^2}.$$

8c

$$\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = v\frac{dv}{dx} = v\cdot\frac{d}{dt}\left(\frac{2}{3}t^{-\frac{2}{3}}\right)\cdot\frac{dt}{dx} = v\cdot\left(-\frac{4}{9}t^{-\frac{5}{3}}\right)\cdot\frac{1}{v} = -\frac{4}{9}\left(\frac{x^3}{8}\right)^{-\frac{5}{3}} = -\frac{4}{9}\left(\frac{x}{2}\right)^{-5} = -\frac{128}{9}x^{-5}.$$