Q17

$P\left(ct,\ \frac{c}{t}\right)$ lies on the rectangular hyperbola $xy = c^2$. The normal at P meets the rectangular hyperbola $x^2 - y^2 = a^2$ at Q and R. Show that P is the midpoint of QR.

Solution:

$\frac{dy}{dt} = -\frac{c}{t^2}$, $\quad \frac{dx}{dt} = c$, $\quad \because \frac{dy}{dx} = -\frac{1}{t^2}$, $\quad \therefore\ m_N = t^2$.

Normal: $y - \frac{c}{t} = t^2(x - ct)$. $\quad y = t^2 x - ct^3 + \frac{c}{t}$.

Substitute into $x^2 - y^2 = a^2$, $\quad x^2 - \left(t^2 x - ct^3 + \frac{c}{t}\right)^2 - a^2 = 0$.

$$x^2 - \left(t^4 x^2 + c^2 t^6 + \frac{c^2}{t^2} - 2ct^5 x + 2ctx - 2c^2 t^2\right) - a^2 = 0. \quad \ldots \quad (1)$$

Let the coefficient of x^2 be α, and $\alpha = 1 - t^4$.

Let the coefficient of x be β, and $\beta = 2ct^5 - 2ct = 2ct(t^4 - 1)$.

Let the constant be γ.

The solution of (1) will be $\quad x = \frac{-\beta}{2\alpha} \pm \frac{\sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}$. $\quad$ which is Q and R.

The midpoint of Q and R has the x coordinate of $\quad \frac{-\beta}{2\alpha} = \frac{2ct(1 - t^4)}{2(1 - t^4)} = ct$, $\quad$ which is the x coordinate of P.

Since QR is a straight line, it follows that P is the midpoint of QR. (QED)