Cambridge 4U p. 103 Q4

P $(a\cos\theta, b\sin\theta)$ lies on the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$. The normal at **P** cuts the x-axis at **X** and the y-axis at **Y**. Show that $\dfrac{\mathbf{PX}}{\mathbf{PY}}=\dfrac{b^2}{a^2}$.

Solution:

Normal: $\dfrac{ax}{\cos\theta}-\dfrac{by}{\sin\theta}=a^2-b^2$.

X : $\left(\dfrac{a^2-b^2}{a}\cos\theta,\ \ 0\right)$.

Y : $\left(0,\ \ -\dfrac{a^2-b^2}{b}\sin\theta\right)$.

$$\mathbf{PX}^2=\left(a\cos\theta-\frac{a^2-b^2}{a}\cos\theta\right)^2+b^2\sin^2\theta=\left(\frac{\cancel{a^2}-\cancel{a^2}+b^2}{a}\right)^2\cos^2\theta+b^4\left(\frac{\sin^2\theta}{b^2}\right)=b^4\left(\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}\right) .$$

$$\mathbf{PY}^2=a^2\cos^2\theta+\left(b\sin\theta+\frac{a^2-b^2}{b}\sin\theta\right)^2=a^4\left(\frac{\cos^2\theta}{a^2}\right)+\left(\frac{\cancel{b^2}+a^2-\cancel{b^2}}{b}\right)^2\sin^2\theta=a^4\left(\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}\right) .$$

$$\frac{\mathbf{PX}^2}{\mathbf{PY}^2}=\frac{b^4\cancel{\left(\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}\right)}}{a^4\cancel{\left(\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}\right)}}=\frac{b^4}{a^4} .$$

$$\therefore\quad \frac{\mathbf{PX}}{\mathbf{PY}}=\frac{b^2}{a^2} .$$