Q7

$P(x_1, y_1)$ is any point on the ellipse: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

a) Find the equation of the tangent at P.

Ans: $\dfrac{x_1 x}{a^2} + \dfrac{y_1 y}{b^2} = 1$.

b) A line is draw from the centre (0,0) parallel to the tangent at P, meets the ellipse at Q. Prove that the area of triangle OPQ is independent of the position of P.

The gradient of the tangent is $\dfrac{y}{x} = -\dfrac{b^2 x_1}{a^2 y_1}$.

Let us find $Q(x, y)$ by subsituting the gradient into the ellipse equation: $b^2 x^2 + a^2 y^2 = a^2 b^2$.

$$b^2 x^2 + a^2 \left(-\frac{b^2 x_1}{a^2 y_1} \cdot x\right)^2 = a^2 b^2, \qquad \cancel{b^2}\left(1 + \frac{b^2 x_1^2}{a^2 y_1^2}\right) x^2 = a^2 \cancel{b^2}, \qquad x^2 = \frac{a^4 y_1^2}{a^2 y_1^2 + b^2 x_1^2} = \frac{a^4 y_1^2}{a^2 b^2}.$$

Subsitute x^2 back to the ellipse for y^2:

$$b^2 \cdot \frac{a^{\cancel{4}\,2} y_1^2}{a^2 y_1^2 + b^2 x_1^2} + \cancel{a^2} y^2 = \cancel{a^2} b^2, \qquad y^2 = b^2 \left(1 - \frac{a^2 y_1^2}{a^2 y_1^2 + b^2 x_1^2}\right), \qquad y^2 = \frac{b^4 x_1^2}{a^2 y_1^2 + b^2 x_1^2} = \frac{b^4 x_1^2}{a^2 b^2}.$$

$$OQ: \quad \sqrt{x^2 + y^2} = \sqrt{\frac{a^4 y_1^2 + b^4 x_1^2}{a^2 b^2}} = \frac{\sqrt{a^4 y_1^2 + b^4 x_1^2}}{ab}.$$

The distance from 0 to P: $\quad d_0 = \left| \dfrac{1}{\sqrt{\left(\frac{x_1}{a^2}\right)^2 + \left(\frac{y_1}{b^2}\right)^2}} \right| = \dfrac{a^2 b^2}{\sqrt{b^4 x_1^2 + a^4 y_1^2}}$

Area of triangle $OPQ = \frac{1}{2} \cdot OQ \cdot d_0 = \frac{1}{2} \cdot \dfrac{\cancel{\sqrt{a^4 y_1^2 + b^4 x_1^2}}}{ab} \cdot \dfrac{a^2 b^2}{\cancel{\sqrt{b^4 x_1^2 + a^4 y_1^2}}} = \frac{1}{2}\, ab$.