Roots of Unity Exercise

Q24

Solve $x^8 + 1 = 0$, express each root in the form $r \text{ cis } \theta$ and then $A + iB$.

Decompose $x^8 + 1$ into real quadratic factors and deduce that
$\cos 4\theta = 8\left[\cos\theta - \cos\frac{\pi}{8}\right]\left[\cos\theta - \cos\frac{3\pi}{8}\right]\left[\cos\theta - \cos\frac{5\pi}{8}\right]\left[\cos\theta - \cos\frac{7\pi}{8}\right]$.

Solution:

By $\omega = \text{cis}\left(\frac{\pi+2k\pi}{8}\right)$, we found 8 roots: $\omega_1, \omega_2, \ldots, \omega_8$.

$\omega_1 = \text{cis}\,\frac{\pi}{8} = \cos\frac{\pi}{8} + i\sin\frac{\pi}{8}$,
$\omega_2 = \text{cis}\,\frac{3\pi}{8} = \cos\frac{3\pi}{8} + i\sin\frac{3\pi}{8}$,
$\omega_3 = \text{cis}\,\frac{5\pi}{8} = \cos\frac{5\pi}{8} + i\sin\frac{5\pi}{8}$,
$\omega_4 = \text{cis}\,\frac{7\pi}{8} = \cos\frac{7\pi}{8} + i\sin\frac{7\pi}{8}$,
$\omega_5 = \text{cis}\,\frac{9\pi}{8} = \cos\frac{9\pi}{8} + i\sin\frac{9\pi}{8} = \cos\frac{7\pi}{8} - i\sin\frac{7\pi}{8} = \overline{\omega_4}$,
$\omega_6 = \text{cis}\,\frac{11\pi}{8} = \cos\frac{11\pi}{8} + i\sin\frac{11\pi}{8} = \cos\frac{5\pi}{8} - i\sin\frac{5\pi}{8} = \overline{\omega_3}$,
$\omega_7 = \text{cis}\,\frac{13\pi}{8} = \cos\frac{13\pi}{8} + i\sin\frac{13\pi}{8} = \cos\frac{3\pi}{8} - i\sin\frac{3\pi}{8} = \overline{\omega_2}$,
$\omega_8 = \text{cis}\,\frac{15\pi}{8} = \cos\frac{15\pi}{8} + i\sin\frac{15\pi}{8} = \cos\frac{\pi}{8} - i\sin\frac{\pi}{8} = \overline{\omega_1}$.

For each of the four conjugation pairs:
$\omega_1 + \omega_8 = \omega_1 + \overline{\omega_1} = 2\cos\frac{\pi}{8}$, $\quad \omega_1 \cdot \omega_8 = \omega_1 \cdot \overline{\omega_1} = 1$,
$\omega_2 + \omega_7 = \omega_2 + \overline{\omega_2} = 2\cos\frac{3\pi}{8}$, $\quad \omega_2 \cdot \omega_7 = \omega_2 \cdot \overline{\omega_2} = 1$,
$\omega_3 + \omega_6 = \omega_3 + \overline{\omega_3} = 2\cos\frac{5\pi}{8}$, $\quad \omega_3 \cdot \omega_6 = \omega_3 \cdot \overline{\omega_3} = 1$,
$\omega_4 + \omega_5 = \omega_4 + \overline{\omega_4} = 2\cos\frac{7\pi}{8}$, $\quad \omega_4 \cdot \omega_5 = \omega_4 \cdot \overline{\omega_4} = 1$.

Because ω_n is a root of $x^8 + 1 = 0$, we can decompose $x^8 + 1$ into real factors, then group them in conjugate pairs:

$$x^8 + 1 = (x - \omega_1)(x - \omega_8) \cdot (x - \omega_2)(x - \omega_7) \cdot (x - \omega_3)(x - \omega_6) \cdot (x - \omega_4)(x - \omega_5)$$

$$= \left[x^2 - (\omega_1 + \omega_8)x + \omega_1\omega_8\right] \cdot \left[x^2 - (\omega_2 + \omega_7)x + \omega_2\omega_7\right] \cdot \left[x^2 - (\omega_3 + \omega_6)x + \omega_3\omega_6\right] \cdot \left[x^2 - (\omega_4 + \omega_5)x + \omega_4\omega_5\right]$$

$$= \left[x^2 - 2\cos\frac{\pi}{8}x + 1\right] \cdot \left[x^2 - 2\cos\frac{3\pi}{8}x + 1\right] \cdot \left[x^2 - 2\cos\frac{5\pi}{8}x + 1\right] \cdot \left[x^2 - 2\cos\frac{7\pi}{8}x + 1\right]$$

Note: $x^8 + 1$ above does not need to be zero; we just make use of the roots of $x^8 + 1 = 0$ to factorise $x^8 + 1$.

Now divide both sides by x^4:

$$x^4 + \frac{1}{x^4} = \left[x - 2\cos\frac{\pi}{8} + \frac{1}{x}\right] \cdot \left[x - 2\cos\frac{3\pi}{8} + \frac{1}{x}\right] \cdot \left[x - 2\cos\frac{5\pi}{8} + \frac{1}{x}\right] \cdot \left[x - 2\cos\frac{7\pi}{8} + \frac{1}{x}\right]$$

Since x is arbitrary, we let $x = \cos\theta + i\sin\theta$, $\quad \frac{1}{x} = x^{-1} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta = \overline{x}$. $\quad x + \frac{1}{x} = x + \overline{x} = 2\cos\theta$.

Likewise, $x^4 = \cos 4\theta$ and $\quad x^4 + \frac{1}{x^4} = 2\cos 4\theta$.

We now have: $\quad 2\cos 4\theta = \left[2\cos\theta - 2\cos\frac{\pi}{8}\right] \cdot \left[2\cos\theta - 2\cos\frac{3\pi}{8}\right] \cdot \left[2\cos\theta - 2\cos\frac{5\pi}{8}\right] \cdot \left[2\cos\theta - 2\cos\frac{7\pi}{8}\right]$

Divide both sides by 2:

$\cos 4\theta = 8\left[\cos\theta - \cos\frac{\pi}{8}\right]\left[\cos\theta - \cos\frac{3\pi}{8}\right]\left[\cos\theta - \cos\frac{5\pi}{8}\right]\left[\cos\theta - \cos\frac{7\pi}{8}\right] \quad \ldots$Q.E.D.